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Non-commutative D- and M-brane Bound States

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Abstract

We analyze certain brane bound states in M-theory and their descendants in type IIA string theory, all involving 3-form or 2-form background fluxes. Among them are configurations which represent NCYM, NCOS and OD p -theories in the scaling limit of OM-theory. In particular, we show how the conditions for the embedding to preserve supersymmetry are modified by the presence of the flux and discuss their relations for the various different bound states. Via the formalism of geometric quantization such a deformation of a supersymmetric cycle is related to a non-commutativity of its coordinates. We also study possible non-commutative deformations of the Seiberg-Witten curve of $\mathcal{N} = 2$ supersymmetric gauge theories due to non-trivial H -flux.

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1 Introduction

Non-commutative Yang-Mills theories (NCYM) arise as effective theories of open strings whose endpoints move on D-branes in the background of a constant antisymmetric tensor field B_{ij} or magnetic field F_{ij} on the branes [1]. The scaling limit which is involved to decouple gravity and massive modes is schematically given by

$$\alpha' \rightarrow 0, \quad F_{ij} \sim \text{finite}.$$

The S-duality of IIB superstrings maps the magnetic background field F_{ij} to electric fields F_{0k} ,

$$F_{0k} = \frac{F_{ij}}{\sqrt{1 + (F_{ij})^2}},$$

and the S-dual of NCYM is usually referred to as NCOS (non-commutative open strings) [2, 3]. The above limit for F_{ij} translates to a critical electric field which implies a theory of light open strings along the $0k$ directions and decoupled from gravity. In order to study non-perturbative configurations like supersymmetric instantons in NCYM, bound states of D0-D($2p$) branes with background magnetic fields were considered, where supersymmetry puts some severe restrictions on the form of the F -field. These systems are T-dual to Dp - Dp' branes intersecting at certain angles which are likewise determined by supersymmetry.

In this paper we will discuss the lifting of such situations to M-theory and explain the relations to OM-theory (open membrane theory) [4, 5]. Therefore we have to study non-threshold bound states of M5- and M2-branes, which are equivalent to M5-branes in the presence of an antisymmetric 3-form field H_{ijk} living on the M5-brane world volume along the direction of the M2-brane. The circle-compactification of the M-theory set-up involves the D4-D2 bound state, the NS5-D2 and the D4-F1 bound states, and the scaling limit of OM-theory maps to the OD2-theory (open D2-brane theory) and to the NCOS limits, respectively.

In the first part of this paper we will analyze the conditions on the fluxes and the shape of these bound states in order to preserve supersymmetry from a ten dimensional perspective. The conditions on the flux F for the three types of bound states in type IIA string theory will be found equivalent to the self-duality condition of H . In particular the supersymmetry conditions on the D4-D2 brane bound state will be investigated in some detail, where the D2-brane is rotated inside the D4-brane by arbitrary angles and, in addition, magnetic fluxes on the D4-brane are turned on. As a result we will show that the D2-brane in the presence of the fluxes describes a deformed 2-cycle with an induced

symplectic form determined by the flux. We then demonstrate by the techniques of geometric quantization that due to the non-trivial symplectic structure the coordinates of the curve that describes the embedding of the D2-brane are non-commutative.

In the second part of the paper we will consider superpositions of two M5-branes, filling different spatial directions, or, in the smooth case, the 2-cycle which is formed by a single M5-brane. The M5-branes will be bound together with two different M2-brane configurations, namely first a separate M2-brane inside each M5-brane, corresponding to M2-branes wrapping the 2-cycle, and second in a fashion such that the intersection of any M2-brane with an M5-brane is a self-dual string. The motivation for these configurations originates from the observation [6] that the embedding of a smooth M5-brane in M-theory for the case of vanishing background fields reproduces the Seiberg-Witten curve of $\mathcal{N} = 2$ supersymmetric gauge theories. Similarly the M-theory lift of certain $\mathcal{N} = 1$ supersymmetric gauge theories leads to a supersymmetric M5-brane whose internal embedding is given by a supersymmetric 3-cycle, i.e. a special Lagrangian submanifold in 6-dimensions [7]. We will see that for the first M2-brane configuration (see table 3) the Seiberg-Witten curve will not be deformed by any of the allowed fluxes, in agreement with some recent discussion on non-commutative $\mathcal{N} = 2$ gauge theories [8].

For the second M2-brane configuration (see table 4), following the work of [9, 10, 11, 12, 13], we have already considered the way a constant H -field affects the geometry of an M5-brane in [14] (see also [15]). We will see that turning on H indeed lifts the Lagrangian condition in the BPS-equations of the M5-brane, which means that the Seiberg-Witten curve in the presence of the flux gets deformed, such that it is no longer holomorphic in the same complex structure. Via geometric quantization this again signals a non-commutativity on the deformed curve. But in the two dimensional case the holomorphicity can be restored by a rotation of coordinates in the definition of the complex structure.

2 Supersymmetric bound states

We first discuss a bound state of a single M5-brane with another single M2-brane inside the M5-brane. From the M5 world volume perspective, an M5-brane with a constant H -field represents a non-threshold bound state of this M5-brane with an M2-brane along two spatial directions of the M5-brane. This will be the prototype and building block of more complicated superpositions we shall encounter later on.

2.1 M5-M2 bound states: OM-theory

In static gauge the embedding of the M5-brane into flat 11d spacetime is realized by a map, which describes the dependence of the transverse coordinates X_i , $i = 6, \dots, 10$ on the brane coordinates x_j , $j = 0, \dots, 5$. Furthermore there is a two-form potential $B_{\mu\nu}$ on the six-dimensional worldvolume of the M5-brane with field strength $H = dB$. A nonlinear self duality constraint is realized on the field H , so that the anti-self-dual part can be computed from the self-dual one:

$$H_{ijk} = \frac{4}{Q} (h_{ijk} + 2(kh)_{ijk}),$$

where k_i^j and Q are defined by $k_i^j = h_{ikl}h^{jkl}$ and $Q = 1 - \frac{2}{3}\text{tr}k^2$. If one considers the case with only H_{012} and H_{345} turned on, this constraint can be solved by parametrizing H_{012} and H_{345} through a single variable, which by convention is chosen to be

$$H_{012} = -\sin(\tilde{\varphi}), \quad H_{345} = \tan(\tilde{\varphi}). \quad (2.1)$$

Equivalently, this can be expressed as

$$H_{012} = -\frac{H_{345}}{\sqrt{1 + (H_{345})^2}}. \quad (2.2)$$

The equations of motion of the M5-brane are obtained from the superembedding approach [16]. For the brane considered here the supersymmetry projector $1 - \Gamma$ is given by

$$\Gamma = \frac{1}{\sqrt{1 + H_{ijk}^2}} \Gamma_{012345} - \frac{H_{ijk}}{\sqrt{1 + H_{ijk}^2}} \Gamma_{ijk}. \quad (2.3)$$

We now consider a bound state of an M5- and an M2-brane along the 012 directions of the world volume of the M5-brane. This is the configuration of table 1. It will be of interest in the decoupling limits of OM-theory, NCOS and OD2-theory, to be explained in the following section.

M5	1	2	3	4	5
M2	1	2			

TABLE 1. M5-M2 bound state configuration

This set-up has been discussed e.g. in [17] and an according gravity solution was constructed in [18]. One has to take into account that the projection operators Γ_{012345}

and Γ_{012} no longer commute, such that the supersymmetric configuration must be a non-threshold BPS bound state. The conserved supersymmetry is determined by the projector

$$\Gamma^{(12)} = \cos(\tilde{\varphi}) \Gamma_{012345} + \sin(\tilde{\varphi}) \Gamma_{012} = \Gamma_{012345} e^{\tilde{\varphi} \Gamma_{345}}. \quad (2.4)$$

It is equivalent to the projector of a single M5-brane with flux given by $H_{012} = -\sin(\tilde{\varphi})$ as in (2.3).

We would now like to address the issue of scaling limits which decouple gravity from the theory on the M5-brane and conserves the influence of the flux H_{012} : OM-theory [5] (see also [19]). While the Planck mass M_p has to go to infinity, the effective scale M_{eff} of a membrane stretching along the worldvolume directions 012 of the M5-brane, which support the flux, can stay finite

$$M_p^3 - \epsilon^{012} H_{012} = M_{\text{eff}}^3 \sim \text{finite}$$

by letting the flux go to its critical value, which implies:

$$H_{012} \sim M_p^3.$$

The resulting theory has light excitations of M2-branes extending in the 012 directions of the M5-brane as dominating degrees of freedom. It has been conjectured that the coordinates of the fluctuating M2-brane are non-commutative, but no proper derivation could be presented up to now.

Upon compactification on circles one can now recover supersymmetric bound states of type IIA branes. The M-theory-IIA dictionary then translates M5-branes to D4-branes or NS5-branes and M2-branes to D2-branes or fundamental F1 strings, respectively. One has three options to pick some direction as the eleventh:

- parallel to only the M5-brane \Rightarrow D4-D2 bound state (NCYM-theory),
- parallel to both of the two branes \Rightarrow D4-F1 bound state (NCOS-theory) or
- orthogonal to both of them \Rightarrow NS5-D2 bound state (OD2-theory).

The 3-form flux becomes an electric 2-form on the D4-branes or a RR 3-form flux on the NS5-branes:

$$R_i H_{ijk} = F_{jk}.$$

Let us go through the different options case by case.

2.2 D4-D2 bound states: NCYM

Compactifying along any of the 345 directions leads to a D4(01234)-D2(012) state. In the appropriate scaling limit $\alpha' \rightarrow 0$ with $F_{34} = R_5 H_{345}$ kept finite the resulting theory is NCYM in (4+1) dimensions [5]. Closed string states as well as massive open string excitations decouple, and one is left with pure YM theory on a non-commutative space. The flat superposition of the two kinds of D-branes is non-supersymmetric. But, in the presence of constant magnetic background fluxes together with non-trivial intersection angles of the two branes, supersymmetry can be restored. In the following we will derive the combined supersymmetry conditions on the fluxes and on the angles from the projector conditions on the spinors.

2.2.1 Projector relations

To be slightly more generic we now consider Dp - Dq -brane bound states, which one can easily specialize to the D4-D2 case. The coordinates are chosen such that the Dp -brane extends into $0, 1, \dots, p$ directions, the Dq -brane into $0, 1, \dots, q$ directions, and $q \leq p$. The Dq -brane can then be rotated inside the world volume of the Dp -brane by any angle φ_{ij} in a plane spanned by x_i and x_j , with $i \leq q$ and $q < j \leq p$. The projector that defines the condition for preserving supersymmetry reads [20]

$$\Gamma_{01\dots p} \epsilon = \tilde{\epsilon} \quad \& \quad \Gamma_{01\dots q} (R\epsilon) = (R\tilde{\epsilon}) \quad \Rightarrow \quad \Gamma_{(q+1)\dots p} R^2 \epsilon = \pm \epsilon, \quad (2.5)$$

where R denotes the rotation

$$R = \exp(\varphi_{ij} \Sigma_{ij}) \quad (2.6)$$

of the spinors, ϵ for left-moving, $\tilde{\epsilon}$ for right-moving supersymmetries and normalizations chosen such that $\Sigma_{ij} = \Gamma_{ij}/2$ takes eigenvalues $\pm i/2$.

Now we also turn on also magnetic fluxes $F_{ij} = \tan(\tilde{\varphi}_{ij})$ on the Dp -brane. Then the supersymmetry condition is identical to the last equation of (2.5), where we have replaced φ_{ij} in R by $\tilde{\varphi}_{ij}$, i.e.

$$\tilde{R} = \exp(\tilde{\varphi}_{ij} \Sigma_{ij}) \quad (2.7)$$

In fact, due to the change of commutation relations, this is now derived from an asymmetric rotation

$$\Gamma_{01\dots q} \epsilon = \tilde{\epsilon} \quad \& \quad \Gamma_{01\dots p} (\tilde{R}\epsilon) = (\tilde{R}^{-1}\tilde{\epsilon}) \quad \Rightarrow \quad \Gamma_{(q+1)\dots p} \tilde{R}^2 \epsilon = \pm \epsilon, \quad (2.8)$$

treating left- and right-moving spinors with opposite phases. This agrees with the fact that the boundary conditions for an open string in the presence of background 2-form

flux can be derived from an asymmetric rotation from pure Neumann or Dirichlet boundary conditions [21].

These conditions for D-branes at angles and D-branes with flux can now be lifted to eleven dimensions, where we restrict ourselves to branes of dimensions less than six. The first case is of course trivial, two D-branes intersecting at any relative angle lift to two M-branes with the same relative angle. For the second, we use the ten-dimensional chirality projector $(1 - \Gamma_{11})$ in order to rewrite the asymmetrically rotated equation in (2.8) as a condition on the 11-dimensional spinor $\eta = \epsilon \oplus \tilde{\epsilon}$

$$\Gamma_{0\dots p} \tilde{R} (1 + \Gamma_{11}) \eta = \tilde{R}^{-1} (1 - \Gamma_{11}) \eta. \quad (2.9)$$

From this we derive

$$\begin{aligned} (1 + \beta - (1 - \beta) \Gamma_{11}) \eta &= \Gamma_{0\dots p} \left(\left(\tilde{R}^2 + \tilde{R}^{-2} \right) + \left(\tilde{R}^2 - \tilde{R}^{-2} \right) \Gamma_{11} \right) \eta \\ &= 2 \Gamma_{0\dots p} (\cos(\tilde{\varphi}_{ij}) + \sin(\tilde{\varphi}_{ij}) \Gamma_{ij} \Gamma_{11}) \eta, \end{aligned}$$

where we have used $\beta = (\Gamma_{0\dots p})^2 = \pm 1$. Obviously, asymmetric rotations or 2-form fluxes on D-branes are lifted to become 3-form fluxes in M-theory. On the contrary, a similar computation shows that the conditions (2.5) lift trivially to eleven dimensions, angles indeed remain angles. All this is in accord with the fact that in M-theory the 2-form field F_{ij} is lifted to H_{ij11} ,

$$R_{11} H_{ij11} = F_{ij}.$$

We have to expect that D-branes with 2-form flux lift to M-branes with 3-form flux simply because there is no 2-form present in M-theory.

Upon compactification on a torus \mathbb{T}^2 the Dirac quantization requires the flux to take discrete values only. Concretely, F_{ij} is rational

$$F_{ij} = \frac{n}{m} \in \mathbb{Q}, \quad \gcd(m, n) = 1, \quad (2.10)$$

where we have set the compact volume to one. The two integers m and n are the D4- and D2-brane charges of the bound state. In the T-dual picture with D2-branes at angles, they translate into the wrapping numbers of these D2-brane on the 1-cycles of the tori [21].

2.2.2 Bound States with supersymmetric field strength

For simplicity consider first a D4-D2 bound state with vanishing angles φ_{ij} and only non-zero fluxes with the components $F_{12} = \tan(\tilde{\varphi}_{12})$ and $F_{34} = \tan(\tilde{\varphi}_{34})$. If there is no flux in the 12 directions, $F_{12} = 0$, one can employ T-duality to get the D2(034)-D0(0) bound state recently considered by [22]. An alternative description was presented in [23], where another T-duality towards a D1(03)-D1(04') bound state of two D1-branes at a relative angle $\phi_{34} = \pi/2 - \arctan(F_{34})$ in the 34-plane was used^{*4}. The condition for the system to preserve some supersymmetry is simply $\phi_{34} = 0$, i.e. $\pi/2 = \arctan(F_{34})$, which translates into an infinite magnetic field F_{34} on the D2(034)-brane. In fact, this configuration then is parallel and trivial. A more generic solution for the D4(01234)-D2(012) system to be supersymmetric is obtained by including relative flux F_{12} as well. The configuration can then be T-dualized into a D2(013)-D2(02'4') state at relative angles $\phi_{12} = \tilde{\varphi}_{12} = \arctan(F_{12})$ and $\phi_{34} = \pi/2 - \tilde{\varphi}_{34} = \pi/2 - \arctan(F_{34})$, primes indicating rotated directions. Then supersymmetry provides the condition

$$F_{12} = -\frac{1}{F_{34}}, \quad (2.11)$$

which translates via $F'_{12} = F_{12}$ and $F'_{34} = -1/F_{34}$ into the familiar self-duality condition $*F' = F'$ of the field strength F' on a T-dual D0-D4 bound state. The non-commutative deformation of the world volume of the D4-brane is ruled by the deformation parameter

$$\Theta_{12} = \frac{1}{F'_{12}} = -\frac{1}{F'_{34}} = -\Theta_{34}. \quad (2.12)$$

In M-theory the magnetic flux is of course lifted as

$$R_5 H_{345} = \tan(\tilde{\varphi}_{34}), \quad (2.13)$$

in agreement with the parametrization in eq. (2.1). An advantage of the D4-D2 bound state as compared to the M5-M2 state is that we have got a completely well defined microscopic description in terms of open string theory for this state. We can use our knowledge about such brane configurations in type IIA to derive some properties of the M-brane state, such as the self-duality (2.1) and the projector conditions (2.4) for preserving supersymmetry in the following.

2.2.3 Supersymmetric cycles with flux

Actually, for the D4(01234)-D2(012) bound state from above there are more possibilities to be supersymmetric than just the flux $F_{12} = -1/F_{34}$. In principle we can introduce

^{*4}In order not to confuse the notation, remember the φ 's denote the rotation angles of the D2-brane inside the D4-brane; on the other hand, the ϕ 's are angles between the D-branes in the T-dual picture. Finally the 'angles' $\tilde{\varphi}$ always parametrize the magnetic fluxes.

any kind of additional constant flux $F_{13}, F_{24}, F_{14}, F_{23}$ on the D4-brane and rotate the D2-brane within the D4-brane by relative angles $\varphi_{13}, \varphi_{14}, \varphi_{23}, \varphi_{24}$. These parameters describe a flat embedding, $f : (x_3, x_4) \rightarrow (x_3 + iX_1(x_3, x_4), x_4 + iX_2(x_3, x_4))$, of the D2-brane into the D4-brane, which in the presence of additional flux will not be a supersymmetric cycle anymore. In [23] the global conditions on angles to preserve any supersymmetry for flat D-branes were generalized to local conditions on branes with non-trivial embedding into space-time. This passing from global to local conditions implies a replacement of the so far flat brane by a generic supersymmetric cycle. The condition are usually phrased in terms of the pull backs of the holomorphic 2-form Ω and of the symplectic form ω (see appendix A), defined for the 2-cycle in question by

$$\begin{aligned} f^*\omega &= (\partial_4 X_1 - \partial_3 X_2) dx^3 \wedge dx^4, \\ f^*\Re \Omega &= (1 - \partial_3 X_1 \partial_4 X_2 + \partial_4 X_1 \partial_3 X_2) dx^3 \wedge dx^4, \\ f^*\Im \Omega &= (\partial_3 X_1 + \partial_4 X_2) dx^3 \wedge dx^4. \end{aligned} \quad (2.14)$$

The conditions for supersymmetry in the absence of fluxes are then simply given by

$$f^*\omega = f^*\Im \Omega = 0. \quad (2.15)$$

In a similar vein we now combine the relations (2.8) and (2.5) for an asymmetric rotation \tilde{R}_1 and a symmetric rotation R_2 respectively to get the conditions for a cycle with flux^{*5}

$$\Gamma_{01234}(\tilde{R}_1 \epsilon) = (\tilde{R}_1^{-1} \tilde{\epsilon}) \quad \& \quad \Gamma_{012}(R_2 \epsilon) = (R_2 \tilde{\epsilon}) \quad \Rightarrow \quad \Gamma_{34} \tilde{R}_1^2 R_2^2 \epsilon = \epsilon. \quad (2.16)$$

Now \tilde{R}_1 and R_2 are no longer commuting such that one cannot remove the deformation by a change of coordinates, as was possible for (2.5) and (2.8) separately. There are three sets of commuting rotations, those simultaneous in 12 and 34, those in 23 and 14 and those in 24 and 13 directions. By decomposing $\tilde{R}_1^2 R_2^2$ into these three components we then get the conditions to preserve any supersymmetry:

$$\begin{aligned} 0 &= F_{12} + \frac{1}{F_{34}}, \\ 0 &= \varphi_{23} - \varphi_{14} + \arctan(F_{23}) - \arctan(F_{14}), \\ 0 &= \varphi_{24} + \varphi_{13} + \arctan(F_{24}) + \arctan(F_{13}). \end{aligned} \quad (2.17)$$

Each line of (2.17) states a condition that is capable to define a flat supersymmetric D-brane bound state by being satisfied globally, whereas they may only be patched together locally. The three rotations, symmetric or asymmetric, corresponds to three different relative $U(1)$ rotation of the two branes. Only together they generate the most general local $SU(2)$ deformation of the globally flat cycle. One may simplify the

^{*5}These conditions can be analysed in full generality.

conditions by choosing coordinates where one of the $U(1)$ rotations is absorbed, such that e.g. $f^*\Im\Omega = 0$. The choice of relative signs in (2.17) is arbitrary and stems from a particular choice of complex structure. It relates the equation with minus signs to the symplectic structure $f^*\omega$ and the one without to $f^*\Im\Omega$. We now follow the usual procedure to replace the flat intersecting branes by a smooth curve $X_1(x_3, x_4)$, $X_2(x_3, x_4)$, which means replacing the global angles φ_{ij} by local quantities according to

$$\tan(\varphi_{ij}) \hat{=} \partial_j X_i. \quad (2.18)$$

Then we find

$$\begin{aligned} \frac{f^*\omega}{f^*\Re\Omega} &= -\frac{F_{23} - F_{14}}{1 + F_{23}F_{14}}, \\ \frac{f^*\Im\Omega}{f^*\Re\Omega} &= -\frac{F_{24} + F_{13}}{1 - F_{24}F_{13}}. \end{aligned} \quad (2.19)$$

These are the conditions that the deformed cycle preserves any supersymmetry in the presence of the 2-form flux on the D4-brane. Note, that in the case $F_{23} = F_{14} = (*F)_{23}$ and $F_{24} = -F_{13} = (*F)_{24}$ the standard conditions (2.15) are recovered. Then, the field strength and the cycle are separately supersymmetric. In (2.19) the deviation of the flux from being self-dual or anti-self-dual is compensated by the deviation of the cycle from being special Lagrangian.

The non-commutativity on the two-dimensional cycle, which describes the embedding of the D2-brane into the D4-brane, due to the fluxes F_{23} and F_{14} is now provided by the non-vanishing of the induced symplectic form $f^*\omega$ in eq. (2.19). As explained in appendix B we can introduce a Poisson bracket via $f^*\omega$ and apply the formalism of geometric quantization to define commutator of the coordinates, i.e.

$$[x^3, x^4] = i\{x^3, x^4\} = i f^*\omega(\text{sgrad } x^3, \text{sgrad } x^4). \quad (2.20)$$

Hence eq. (2.19) suggests a noncommutative deformation of the operators which are associated to the coordinates of the cycle that describes the embedding of the D2-brane into the D4-brane by the presence of additional 2-form flux. The flux $F_{12} = -1/F_{34}$ would not have been sufficient to make this construction. In what sense this non-commutativity arising from (2.19) can be understood from a microscopic point of view, and how the values of the non-commutativity parameters can be reconciled with the Θ^{ij} parameters known for NCYM, remains an open question.

Whenever the flux of the D4-D2 system is not tuned in the supersymmetric fashion, there will appear a tachyon in the open string spectrum of strings stretching between the two branes. It is believed that this signals a condensation mechanism towards the

true ground state of the system, which is again BPS. At the critical point of the field strength, when the tachyon becomes massless, one suspects a marginal deformation that takes the bound state with tachyon condensate into the state described by (2.19). This is only a special case of the more general scenarios of Dq-Dp bound states considered in [22, 23, 24, 25].

2.3 D4-F1 bound states: NCOS

The M2-M5 configuration also allows compactification in 12 directions, say we take 2, to a D4-F1 bound state. It is related by a chain of a T-duality along any of the 345 directions, e.g. 5, the S-duality of type IIB and another T-duality along 5 to the D4-D2 bound state discussed above [17] after only exchanging the labels for the 2 and 5 directions (see also [26, 27]). The dualities also relate the conditions for preserving supersymmetry by just erasing Γ 's and switching chiralities appropriately. The mapping of the magnetic flux F_{34} to an electric flux F_{01} has been given in [3] by

$$F_{01} = \frac{F_{34}}{\sqrt{1 + (F_{34})^2}} = - \frac{1}{\sqrt{1 + (F_{12})^2}} = - \sin(\tilde{\varphi}), \quad (2.21)$$

via the condition eq. (2.11) imposed by supersymmetry of the D4-D2 state on the F -flux. A formula for the Dirac quantization of the flux on a Dp-F1 bound states has also been derived in [28] by boundary state techniques. It reproduces the flux quantization (2.10) on the D4-D2 bound state. Dual gravity solutions have been constructed in [29, 30]. Lifting the flux to eleven dimension gives the electric component

$$R_2 H_{012} = F_{01} = - \sin(\tilde{\varphi})$$

of the 3-form. Together with (2.21) this precisely reproduces the condition (2.1) on H after scaling the radii to 1. In other words, the supersymmetry of the type IIA D4-D2 bound state implies the self-duality of H .

It has been shown in [3] that the OM scaling limit of the M5-M2 bound state reduces to the NCOS limit on a D4-brane provided the radius of the circle and the flux is tuned in a particular way:

$$R_2 = G_o^2 \sqrt{\alpha'_{\text{eff}}}, \quad M_{\text{eff}}^3 = \frac{1}{2 G_o^2 \alpha'^{3/2}_{\text{eff}}}. \quad (2.22)$$

Then G_o stands for the open string coupling and α'_{eff} for the effective scale of fundamental strings. In NCOS the closed string excitations decouple from the brane but all open string

states remain at finite mass. The electric field introduces a non-commutativity of the time-space coordinates 01, governed by the parameter

$$\Theta_{01} = \Theta_{34} G_o^2 \quad (2.23)$$

This limit has been introduced as the weakly coupled S-dual of the strongly coupled NCYM-theory that arises on a D-brane in the presence of a magnetic 2-form flux, when going to strong coupling.

2.4 NS5-D2 bound states: OD2-theory

By compactifying along any of transverse directions one obtains a D2-brane inside an NS5-brane with a RR-flux C_{ijk} inherited from H_{ijk} . This state is related via a T-duality in 3 direction, an S-duality which trades the NS5-brane for a D5-brane and another T-duality on 3 to the D4-D2 bound state discussed above, after only exchanging the labels 3 and 5. This chain of dualities transforms the 3-form C_{345} on the NS5-brane into the magnetic flux F_{34} on the D4-brane. In [31, 30] the gravity solution for such a bound state has been calculated. From the formulas for the RR-flux on the NS5-D2 bound state given there, one can read off the coefficients

$$C_{012} = -\sin(\tilde{\varphi}), \quad C_{345} = \tan(\tilde{\varphi})$$

The Dirac quantization again forces $\tan(\tilde{\varphi})$ to be rational. Together, the NS5-D2 bound state also carries the information of the self-duality of H .

By tuning the radius and the flux of the resultant OM-theory formally in the same way as in (2.22)

$$R_6 = G_{o2}^2 \sqrt{\tilde{\alpha}'_{\text{eff}}}, \quad M_{\text{eff}}^3 = \frac{1}{2 G_{o2}^2 \tilde{\alpha}'_{\text{eff}}{}^{3/2}}, \quad (2.24)$$

one gets the so-called OD2-theory with light open D2-branes excitations on the world volume of the NS5-brane but closed strings decoupled again. G_{o2} and $\tilde{\alpha}'_{\text{eff}}$ are now the D2-brane coupling constant and the effective scale of fundamental strings.

3 Superpositions of M-brane bound states

In this section we study two types of superpositions of M2-M5 bound states. The basic building block in both cases is the Kähler calibration of two M5-branes as given in table 2.

M5	1	2	3	4	5		
M5'			3	4	5	6	7

TABLE 2. Kähler calibration

It consists of two intersecting M5-branes, whose embedding into the \mathbb{R}^4 spanned by the 1267 directions is the holomorphic cycle that governs the dynamics of the $\mathcal{N} = 2$ gauge theory in the four non-compact space-time directions 0345 [6]. In the following, this setting is then combined with additional M2-branes. First, in the way of the previous chapters, a separate M2-brane inside each M5-brane, and second in a fashion such that the intersection of any M2-brane with an M5-brane is a self-dual string. The first is a direct generalization of the previous chapters. It will not lead to a deformation of the Seiberg-Witten curve, whereas the second case does.

3.1 M2-branes wrapping the Seiberg-Witten curve

Because the respective projectors (2.4) mutually commute the bound states that have been studied so far as isolated states can also be superposed:

$$[\Gamma^{(12)}, \Gamma^{(67)}] = 0.$$

Such a configuration is the set-up of table 3.

M5	1	2	3	4	5		
M2	1	2					
M5'			3	4	5	6	7
M2'						6	7

TABLE 3. Bound state superposition

It may be interpreted as an M2-brane wrapping the Seiberg-Witten curve, or, equivalently, as a Seiberg-Witten curve with additional 3-form flux turned on.

In general, a compactification down to type IIA now leads to a superposition of either NS5-D2 and D4-F1 or of two D4-D2 bound states. Along the compact eleventh direction the configuration looks like F1 and D4, with 2-form flux on the D4 along the world volume of the F1, and transverse to it, like D2 and NS5 with 3-form flux on the NS5 along the worldvolume of the D2-brane, which has been depicted in figure 1.

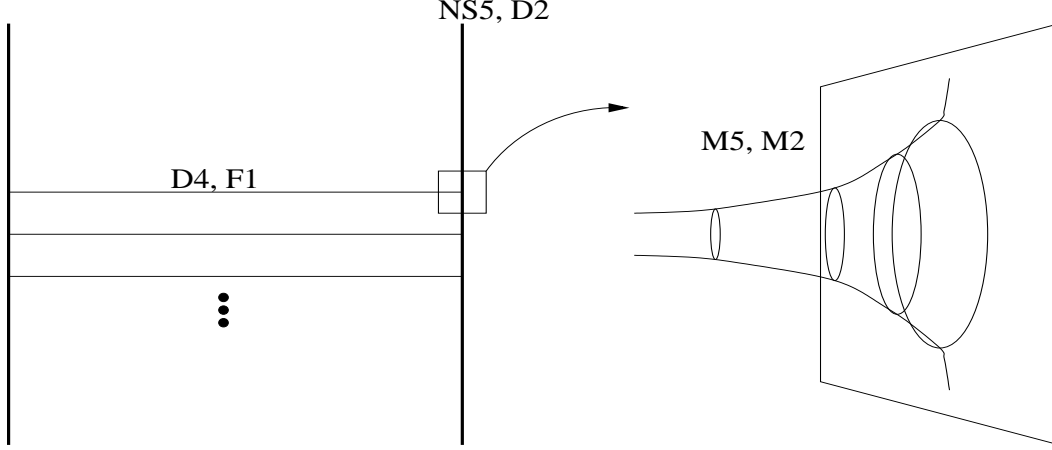


Fig. 1 Blowing up the eleventh direction

The scaling limit of OM-theory now applied to the superpositions of bound states then reveals a superposition of such effective theories of noncommutative open D2-branes and fundamental strings. Upon compactification along any of the 1267 directions we get NCOS along the eleventh direction and OD2 transverse to it. Interestingly, both, the NCOS and the OD2 scaling limits (2.22) and (2.24) are compatible: Given the radius R_{11} and the critical scaling limit of the flux, i.e. M_{eff} , the couplings and, as well, the fundamental string scales are identical for both, the OD2 fluctuations on the NS5-brane, as well as the open string fluctuations on the D4-brane. This had to be expected as both stem from the fluctuations of M2-branes in the eleven-dimensional OM-theory. While a complete description of the (NS5,D4,D2,F1) bound state and its scaling limit is not known to us, partial results for such bound states have for instance been discussed in [31, 32]. Their gravity duals of bound states of the kind (NS5,D4,D2) and (D4,F1) have been derived and the NCOS or OD2 limits of such systems have been analyzed.

Let us again look closer at the limit which we know most about, a compactification along the 5 direction towards a D4-D2-D4'-D2' bound state. We call the field strength on the first D4-D2 bound state F , on the second F' and have the equations (2.19) for both states separately after adapting indices. The Seiberg-Witten curve is obtained by analyzing the embedding of X_6 and X_7 as functions of x_1 and x_2 . If we now write symmetric and asymmetric rotation operators for any kind of angle and flux which can occur for the state of table 3, we have to realize that there are no fluxes that could deform

the Seiberg-Witten curve. Deforming the equations

$$\begin{aligned}\varphi_{16} + \varphi_{27} &= 0 \Rightarrow f^*\omega = 0, \\ \varphi_{17} + \varphi_{26} &= 0 \Rightarrow f^*\Im \Omega = 0\end{aligned}\tag{3.1}$$

would require fluxes F_{ij} , $i \in \{1, 2\}$, $j \in \{6, 7\}$, which are not present here. This situation is very similar to that of Dp - Dp' bound states in [23]. All the other equations which can be derived analogously to (2.19) decouple, and are identical to those of an isolated D4-D2 bound state. Thus, we have to conclude that the supersymmetric cycle, which describes the embedding of the D4'-brane relative to the D4-brane is not deformed and remains special Lagrangian. Such we have to expect that also in M-theory an M2-brane that wraps the Seiberg-Witten curve does not lead to any non-commutative deformation of the curve, which should otherwise be inherited by the type IIA realization. This result is supported by [8], where non-commutative $\mathcal{N} = 2$ gauge theories were recently studied by more direct means with the result that the Seiberg-Witten curve is kept unchanged.

3.2 Self-dual strings on an M5-brane

Finally we like to consider another configuration of M5- and M2-branes, where the M2-branes intersect the world volume of the M5-branes in self-dual strings. It has been shown to describe $\mathcal{N} = 2$ supersymmetric gauge theory in the presence of BPS monopoles whose positions are given by the positions of the M2-branes in the space-time directions 0345.

$\overline{\text{M5}}$	1	2	3	4	5		
M5			3	4	5	6	7
M2	1					6	
M2		2					7

TABLE 4. Brane configuration

Now the M2-brane end as points on either of the two asymptotic M5-branes. We want to study again the embedding of the two-dimensional curve in the directions 1267, which is described by the embedding map $f : (x_1, x_2) \rightarrow (x_1 + iX_6(x_1, x_2), x_2 + iX_7(x_1, x_2))$. Unlike the previous M5-M5' bound state the 2-cycle will be now deformed by the presence of the H -field. The derivation of the BPS-equations is identical to the derivation in [14] or the more general discussion in [34]. The tangent frame the non zero components of H can also be expressed in terms of h_{012} . The two relevant components of eq. (2.1) read:

$$H_{012} = -\sin \tilde{\varphi} = \frac{4h_{012}}{4h_{012}^2 + 1}, \quad H_{345} = \tan \tilde{\varphi} = \frac{4h_{012}}{4h_{012}^2 - 1}. \quad (3.2)$$

The spacetime frame components are distinguished from them by an additional twiddle \tilde{H}_{012} . The BPS-equations (compare with eq. 2.19) read

$$[f^*\omega] = \tilde{H}_{012} = -\sin \tilde{\varphi} \sqrt{-g}, \quad (3.3)$$

$$[f^*\Re \Omega] = \frac{\tilde{H}_{012}}{\tilde{H}_{345}} = -\cos \tilde{\varphi} \sqrt{-g}, \quad (3.4)$$

$$[f^*\Im \Omega] = 0. \quad (3.5)$$

Equation (3.3) is a gauged symplectic structure, i.e. it is proportional to H_{012} . Again, using the methods of geometric quantization, it leads to a non-commutativity on the deformed cycle. The tangent frame component H_{012} can be easily interpreted with respect to the Grassmanian $G(2,4) = S_+^2 \times S_-^2$, defined in appendix A. The algebraic identity

$$\det g = [f^*\Im \Omega]^2 + [f^*\Re \Omega]^2 + [f^*\omega]^2. \quad (3.6)$$

provides a parametrisation of the sphere S_+^2 by:

$$\begin{aligned} -[f^*\omega]/\sqrt{g} &= \sin \tilde{\vartheta} \sin \tilde{\varphi}, \\ -[f^*\Re \Omega]/\sqrt{g} &= \sin \tilde{\vartheta} \cos \tilde{\varphi}, \\ -[f^*\Im \Omega]/\sqrt{g} &= \cos \tilde{\vartheta}. \end{aligned} \quad (3.7)$$

Obviously eq. (3.5) leads to $\tilde{\vartheta} = \pi/2$ and we can identify the component H_{012} with an angle of S_+^2 by:

$$H_{012} = -\sin \tilde{\varphi}.$$

As explained at the end of appendix A this can be interpreted as a brane rotation if one compares with the standard choice of holomorphic coordinates.

A Notation

The definition of the BPS-solutions can be restated in terms of certain closed p-forms (calibrations), which are defined after fixing a complex structure on the embedding space.

Since we are mainly concerned with 2-cycles in \mathbb{R}^4 , the space of q^1, p^1, q^2, p^2 , the set of complex structures is given by the set of matrices

$$J = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{pmatrix}$$

with (a_1, a_2, a_3) a point on a sphere of unit length. If we select one of them, say $J_0 = (1, 0, 0)$, the complex coordinates are $z^1 = q^1 + i p^1$ and $z^2 = q^2 + i p^2$.

In these complex coordinates the sphere of complex structures may be identified with the sphere of selfdual 2-forms in \mathbb{R}^4 . Then the coordinates (a_1, a_2, a_3) refer to the basis below:

$$\begin{aligned} \omega &= \frac{i}{2} (dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2), \\ \Re \Omega &= \Re (dz^1 \wedge dz^2), \\ \Im \Omega &= \Im (dz^1 \wedge dz^2). \end{aligned} \tag{A.1}$$

The tangent planes of a generic two manifold in \mathbb{R}^4 are in one to one correspondence with the space of two planes in \mathbb{R}^4 , $G(2, 4)$, which can be identified with the manifold $S_+^2 \times S_-^2$. Each of the two spheres is embedded in the space of self- or anti-self-dual 2-forms, respectively. If one identifies the sphere of complex structures on \mathbb{R}^4 with one of the S_\pm^2 , say S_+^2 (the difference is only the choice of orientation of \mathbb{R}^4), the calibration condition restricts the remaining planes to $J_0 \times S_-^2$. This reflects the fact, that the Kähler cycle is the one where the symplectic form becomes maximal. In principle one can compute the complex combination for each point in S_+^2 . In section 3.2 we are concerned with the following pencil of complex structures

$$(a_1, a_2, a_3) = (-\sin \tilde{\varphi}, -\cos \tilde{\varphi}, 0) \tag{A.2}$$

with the complex coordinates

$$\begin{aligned} \tilde{z}^1 &= q^1 - i (\sin \tilde{\varphi} \cdot p^1 + \cos \tilde{\varphi} \cdot q^2) \\ \tilde{z}^2 &= p^2 - i (\cos \tilde{\varphi} \cdot p^1 - \sin \tilde{\varphi} \cdot q^2). \end{aligned}$$

This could be seen as a rotation inside the $q^2 - p^1$ -plane.

The projections of the Kähler form in the rotated complex coordinates \tilde{z}^1, \tilde{z}^2 to the old basis in eq. (A.1) leads to the equations (3.3) - (3.5).

B Definition of the commutator

Here we collect all the details necessary to explain the transition from the Poisson bracket to the commutator of coordinates. The construction uses essentially a symplectic structure ω . In normal form it reads $\omega = \sum_i dq^i \wedge dp^i$. Here q^i and p^i are the conjugated variables. To define a Poisson bracket we need the symplectic gradient $\text{sgrad } g$ of a function g . It is simply defined by

$$\text{sgrad } g = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \cdot \text{grad } g.$$

Now the Poisson bracket of two functions g and h is defined via ω by the formula:

$$\{g, h\} = -\omega(\text{sgrad } g, \text{sgrad } h).$$

For the symplectic structure above this leads to the standard relations

$$\{x^i, p^j\} = \delta^{ij}. \quad (\text{B.1})$$

This is a realisation of the usual Heisenberg algebra \mathcal{A}_H .

So far the construction is standard. Quantization can be performed by applying a construction called geometric quantization [35]. Given a symplectic manifold (M, ω) it constructs a linear map $\rho : \mathcal{A} \rightarrow \mathcal{O}$ from an algebra \mathcal{A} of functions on M to the set of hermitian operators \mathcal{O} on a Hilbert space^{*6} $\mathcal{H}(M)$, satisfying the following constraints:

$$\rho(1) = \mathbb{1}_{\mathcal{H}}, \text{ if } 1 \in \mathcal{A} \quad (\text{B.2})$$

$$\rho(F^*) = \rho(F)^\dagger \quad (\text{B.3})$$

$$[\rho(F), \rho(G)] = -i\hbar \rho(\{F, G\}) \quad \forall F, G \in \mathcal{A} \quad (\text{B.4})$$

On a patch $U_\alpha \subset M$ the map ρ is given by

$$\rho_\alpha(F) = -i\hbar \text{sgrad } F + F - \theta_\alpha(\text{sgrad } F) \quad (\text{B.5})$$

with $\theta_\alpha = \sum_{k=1}^n p_k dq_k$ and $\omega = d\theta_\alpha$. The consistency of the construction leads to the Bohr-Sommerfeld quantization:

$$\frac{1}{2\pi\hbar} \int_{2\text{-cycle}} \omega \in \mathbb{Z}. \quad (\text{B.6})$$

^{*6}The technical difficulty is the precise definition of the Hilbert space. Usually the space of functions on M is too large.

Applying the procedure outlined before to the case at hand by combining eq. (B.1) and eq. (B.4) leads to

$$[\rho(x^i), \rho(p^j)] = -i\hbar\delta^{ij}. \quad (\text{B.7})$$

with \hbar a free constant (here $\hbar = 1$).

The formula eq. (B.6) is the first Chern number of the line bundle with curvature ω . Its physical interpretation is the following. By the S^1 -compactification the M2- and M5-branes are reduced to a fundamental string or D4-brane, respectively. Each of these branes carries a winding number, one of which classifies the magnetic the other one the electric charge of the system. The magnetic charge is precisely the number above. The electric charge is invisible in our geometric setup. The reason is quite simple. Since we are dealing only with a single electrically charged object, there is no possibility to define what is meant by the minimal quantum of electrical charge.

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